Relativistic three–dimensional two– and three–body equations on a null plane and applications to meson and baryon Regge trajectories

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We start from a field-theoretical model of zero range approximation to derive three-dimensional relativistic two- and three-body equations on a null plane. We generalize those equations to finite range interactions. We propose a three-body null-plane equation whose form is different from the one presented earlier in the framework of light-cone dynamics. We discuss the choices of the kernels in two- and three-body cases and apply our model to the description of meson and baryon Regge trajectories. Our approach overcomes some theoretical and phenomenological difficulties met in preceding relativized treatments of the three-body problem.

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I. INTRODUCTION

It is very well known that nonrelativistic potential models can be successfully applied to the description of the heavy $q\overline{q}$ spectra (see ref. [1] and the recent review [2]). However when the states containing light quarks (u,d,s) are considered relativistic effects have to be taken into account (see, e.g. [3–6]). Relativistic few body problem has received a great attention in hadronic and nuclear physics and a lot of papers have been devoted to the problem (see e.g. reviews [7–9] and references therein). Different methods for deriving relativistic few-body three-dimensional equations have been discussed. Some authors use a diagrammatic approach, i. e., they select some leading diagrams and project them onto the three-dimensional momentum space. Others make use of effective Hamiltonians, employing various assumptions for their constructions.

The most appealing are the relativistic approaches based on the null-plane (or light-cone) dynamics or, equivalently, on the analysis of Feynman diagrams in the infinite momentum frame (IMF) (see recent papers [10,11] and references therein). One of the main advantages of these approaches is the very well known fact that the wave function of the bound state has a simplest form in IMF, where pair creation from vacuum is suppressed. This was stressed a long time ago by Weinberg [12]. Indeed in the covariant representation Lorentz invariance of the S matrix at any order of perturbation theory is ensured by the contributions containing different numbers of particles and antiparticles in intermediate states. At the same time diagrams of noncovariant perturbation theory in IMF are dominated by the simplest intermediate states where all the particles have positive energies.

To make calculations realistic we must restrict the number of degrees of freedom. This is usually done by imposing cutoff on the number of particles and considering only $q\overline{q}$ sector

in the case of mesons and $3q$ sector in the case of baryons. In order to derive a two-body null plane equation in the three-dimensional form, one can use a Hamiltonian approach in which the null planes $\{x_- = \frac{1}{\sqrt{2}}\,(x^0 + x^3) = const\}$ play the role of equal time surfaces [7,10,11,13], or eliminate relative null plane time x_- in the covariant approaches [7,9,15–17]. However in such approaches it is not trivial to define the angular momentum operators [7,9,13]. This problem is circumvented in the papers which use the Relativistic Hamiltonian Dynamics (RHD) in the null plane form (or Null Plane Dynamics, NPD, or even Light Cone Dynamics, LCD), where the Poincaré generators for the system of two or three interacting particles are directly constructed in terms of internal variables describing a system (see [7–9,13]).

Generally there is no exact correspondence between the results based on RHD or NPD and quantum field theory (QFT). Such a correspondence can be found in the Zero Range Approximation (ZRA), where the relative time can be explicitly eliminated and two-body three-dimensional null-plane equation can be derived from the Bethe-Salpeter equation [9,17]. In this paper we discuss a generalisation of this equation to finite range interactions, which is equivalent to a representation of NPD, where the angular momentum operators have the same form as for free particles [13], so that, under rather general assumptions, covariance [7] of the theory is guaranteed. Although our results on the two-body null plane equation are not new, the discussion we present here gives new insight on the relation between NPD and QFT. Moreover we relate the kernel in the relativistic two-body null plane equation to the potential used by Godfrey and Isgur [3]; this, together with linearity of meson Regge trajectories, suggests that for large separations between constituents the interaction should be of the oscillator type, whose parameter is proportional to the string tension squared.

Such a kernel leads to predictions on Regge trajectories for mesons composed of light (u, d and s) quarks which are in good agreement with data, including small deviations from linearity for strange mesons.

Another new point in our paper is the derivation of three-body null plane equations. Starting from the covariant three-body equation, we show that in ZRA two relative times can be eliminated and a three-dimensional three-body null plane equation for bound or scattering cases can be derived. We consider also a simple way of generalizing our treatment to finite range interactions, in line with Relativistic Hamiltonian Dynamics.

The form of the three-body null plane equations is different from the one proposed earlier in the framework of RHD [18], as well as from the naive relativistic generalization of the Schrödinger equation used in refs. [3,4,19] for the description of baryon spectra. In particular, as we show in the present paper, our approach presents theoretical and also phenomenological advantages over these treatments.

We apply three-body null plane equations to Regge trajectories for a system of three relativistic quarks, analyzing the kernel corresponding to a three-body force with string junction, involved in the mass operator squared. We take this operator in a form which commutes with angular momentum in a representation which coincides with the one for free particles, so that also the three body equation yields a covariant [7] description of dynamics. Approximate linearity of Regge trajectories fixes the dependence of the kernel on two different relative coordinates, which for large relative separations has to be still of the oscillator type. Equating the slopes of two Regge trajectories corresponding to the orbital excitations of two relative degrees of freedom fixes the sector that describes the relativistic recoil of the moving two-quark subsystem. The implementation of the condition that the slope of the

diquark-quark orbital excitations should be same as for meson Regge trajectories fixes the oscillator parameter. We apply the model to the baryon Regge trajectories composed from the light quarks and show that the predictions of our model are in good agreement with data, reproducing small deviations from linearity.

It is worth stressing that, unlike the approaches based on a relativized Schrödinger equation, our three-body Hamiltonian satisfies cluster separability for two-body forces; furthermore, implementing our model with three-body interactions allows to pass from mesons to baryons, without re-adjusting fundamental parameters or assuming a different kind of interaction between quarks.

The paper is organized as follows. In Sect. 2 we carry on the program of studying the relativistic two-body equation on a null plane. Starting from the covariant Bethe-Salpeter equation and from a field theory model in ZRA, we derive a relativistic two-body equation for a null-plane wave-function. Then we generalize this equation for finite range interactions and introduce the relativistic null-plane Lippmann- Schwinger equation.

In Sect. 3 we focus on the three-body equation. As in the two-body case, we start from ZRA; then, using the form of the equation introduced for finite range interaction in the two-body case, we formulate a relativistic equation for a three-body system appropriate for this kind of interactions. Moreover we show that this equation satisfies cluster separability condition.

In Sects. 4 and 5 we apply two- and three-body equations to the study of meson and baryon Regge trajectories. We discuss the choice of the kernels in two- and three-body cases and we apply the model to the description of the experimental data of meson Regge trajectories composed of $q\overline{q}$, $q\overline{s}$ (or $s\overline{q}$) and $s\overline{s}$ orbital excitations and Regge trajectories of

baryons composed of u- and/or d-quarks. We consider also the Λ Regge trajectory.

Sect. 6 is devoted to conclusions.

II. TWO - BODY NULL PLANE EQUATION

Let us consider the case of two non-identical, spinless interacting particles, with masses m_1 and m_2 respectively. The Bethe-Salpeter (BS) equation for the reduced amplitude χ reads [20,21]

$$(p_1^2 - m_1^2)(p_2^2 - m_2^2)\chi(p_1, p_2) + \int \frac{d^4p_1'}{(2\pi)^4}\frac{d^4p_2'}{(2\pi)^4}(2\pi)^4\delta^4(P - p_1' - p_2')U_{12}\ (p, p'; P)\chi(p_1', p_2') = 0, \tag{2.1}$$

where

$$p = \eta p_1 - (1-\eta)p_2, \qquad p' = \eta p_1' - (1-\eta)p_2' \qquad (0 < \eta < 1) \tag{2.2}$$

and U_{12} is the kernel in momentum space.

The BS equation meets some intrinsic difficulties. First of all, the normalization condition for χ

$$\int \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4}\overline{\chi}(p_1, p_2)\frac{d}{dP_\mu}[(p_1^2 - m_1^2)(p_2^2 - m_2^2)(2\pi)^4\delta^4(p' - p) + U_{12}(p, p'; P)]\chi(p_1, p_2) = 2P_\mu \tag{2.3}$$

depends on the interaction through the kernel $U_{12}(p, p'; P)$, which is a source of complications.

Moreover χ depends on the relative energy, therefore $\chi_P(x)$ depends on the relative time t. This property of the reduced amplitude χ leads to different complications when we try to apply the BS equation to physical problems. Indeed, in the ladder approximation the

BS approach does not exhibit the correct static limit, i. e., when, e. g., m_1 becomes large, while keeping m_2 fixed: some solutions exist with negative norm or which do not satisfy the correct non-relativistic limit. This problem can be avoided by eliminating the relative time dependence [9,17].

Let us define a null plane projection of the BS amplitude as follows:

$$\psi(\mathbf{p}_1, \mathbf{p}_2) = \int dp_{1-} dp_{2-} p_{1+} p_{2+} \delta(P_- - p_{1-} - p_{2-}) \chi(p_1, p_2) \ , \tag{2.4}$$

where $\mathbf{p}_i$ are null plane three-momenta, i. e.,

$$\mathbf{p}_i \equiv (\mathbf{p}_{iT}, p_{i+}) \tag{2.5}$$

and

$$p_{i+} = \frac{E_i + p_{iz}}{\sqrt{2}}, \quad \mathbf{p}_{iT} \equiv (p_{ix}, p_{iy}) \ . \tag{2.6}$$

We assume the kernel of eq. (2.1) to be independent of relative null-plane energy p_-, i. e.,

$$U_{12} = U_{12}(\mathbf{p}, \mathbf{p}'; P), \tag{2.7}$$

where

$$\mathbf{p} = \eta \mathbf{p}_1 - (1-\eta)\mathbf{p}_2, \qquad \mathbf{p}' = \eta \mathbf{p}'_1 - (1-\eta)\mathbf{p}'_2. \tag{2.8}$$

This amounts to assuming an instantaneous interaction in the null plane time x_+. Then the reduced BS amplitude may be rewritten as

$$\chi(p_1, p_2) = -\frac{1}{\pi} \frac{\int d\Gamma_{12} U_{12}(\mathbf{p}, \mathbf{p}'; P) \psi(\mathbf{p}'_1, \mathbf{p}'_2)}{(p_1^2 - m_1^2)(p_2^2 - m_2^2)}, \tag{2.9}$$

where

$$d\Gamma_{12} = \frac{1}{2(2\pi)^3}\frac{d\mathbf{p}'_1}{p'_{1+}}\frac{d\mathbf{p}'_2}{p'_{2+}}\delta^3(\mathbf{P} - \mathbf{p}'_1 - \mathbf{p}'_2). \tag{2.10}$$

Integrating both sides of (2.9) over $p_{1+}p_{2+}dp_{1-}dp_{2-}\delta(P_- - p_{1-} - p_{2-})$, we derive the following equation for a null plane wave-function:

$$\psi(\mathbf{p}_1, \mathbf{p}_2) = iP_+G_0(x_1, \mathbf{p}_{1T}; x_2, \mathbf{p}_{2T})\int d\Gamma_{12}U_{12}(\mathbf{p}, \mathbf{p}'; P)\psi(\mathbf{p}'_1, \mathbf{p}'_2), \tag{2.11}$$

where

$$G_0(x_1, \mathbf{p}_{1T}; x_2, \mathbf{p}_{2T}) = -[\frac{\mathbf{p}^2_{1\mathrm{T}} + \mathrm{m}^2_1}{x_1} + \frac{\mathbf{p}^2_{2\mathrm{T}} + \mathrm{m}^2_2}{x_2} - M^2 - \mathbf{P}^2_T - i\epsilon]^{-1} \tag{2.12}$$

and

$$M^2 = P^2, \qquad x_i = \frac{p_{i+}}{P_+}, \qquad \mathbf{P}_T = \mathbf{p}_{1T} + \mathbf{p}_{2T}. \tag{2.13}$$

Moreover let us assume the kernel U_{12} to be of the type

$$U_{12} = U_{12}(x, \mathbf{p}_T; x', \mathbf{p}'_T)\ , \tag{2.14}$$

where

$$x = \eta x_1 - (1-\eta)x_2, \qquad \mathbf{p}_T = \eta\mathbf{p}_{1T} - (1-\eta)\mathbf{p}_{2T}\ . \tag{2.15}$$

Then we may define a null-plane wave-function Ψ that, according to (2.11), satisfies the following equation:

$$\begin{aligned}&[\frac{\mathbf{p}^2_{1T} + m^2_1}{x_1} + \frac{\mathbf{p}^2_{2T} + m^2_2}{x_2} - M^2 - \mathbf{P}^2_T]\Psi(x_1, \mathbf{p}_{1T}; x_2, \mathbf{p}_{2T}) + \\ &\qquad i\int dL'_{12}U_{12}(x, \mathbf{p}_T; x', \mathbf{p}'_T)\Psi(x'_1, \mathbf{p}'_{1T}; x'_2, \mathbf{p}'_{2T}) = 0,\end{aligned} \tag{2.16}$$

where

$$dL'_{12} = \frac{1}{2(2\pi)^3}[\prod_{r=1}^{2}\frac{dx'_r}{x'_r}d\mathbf{p}'_{rT}]\delta(1 - x'_1 - x'_2)\delta^2(\mathbf{P}_T - \mathbf{p}'_{1T} - \mathbf{p}'_{2T}). \tag{2.17}$$

At this point we discuss approximation (2.7). The simplest case is ZRA, i. e., when U_{12} is a constant. In this case the null plane wave-function reads

$$\Psi = \frac{N}{\frac{\mathbf{q}_T^2 + m_1^2}{\xi} + \frac{\mathbf{q}_T^2 + m_2^2}{1-\xi} - M^2 - i\varepsilon}, \tag{2.18}$$

where we have introduced the variables

$$\xi = x_1, \qquad \mathbf{q}_T = (1-\xi)\mathbf{p}_{1T} - \xi \mathbf{p}_{2T} \tag{2.19}$$

and the constant N is determined by the normalization condition (2.3), i. e.,

$$\frac{1}{2(2\pi)^3} \int \frac{d\xi d\mathbf{q}_T}{\xi(1-\xi)} |\Psi(\xi, \mathbf{q}_T)|^2 = 1, \tag{2.20}$$

which turns out to coincide with the usual normalization of the infinite momentum frame wave-function.

In this limit a null plane description is equivalent to field theory; however it has limited applications. It is therefore important to construct kernels of finite size, using different models. For example, it is possible to use ladder approximation to the BS equation and to project it on the null plane. Such an equation was suggested by Weinberg [12]. Another example was considered by t'Hooft [15], who derived a two-body null plane equation for the $q\overline{q}$ system in $1+1$ QCD, taking the limit of $N_c \to \infty$, in such a way that $\alpha_s N_c$ have a finite, constant value.

A great deal of papers were devoted to the analysis of different aspects of Hamiltonian null plane approach [10]. Here we try to impose the Tamm-Dancoff cutoff on the number of particles, while conserving a very simple representation of angular momentum operators. Generally speaking, in null plane formalism angular momentum operators should depend on interaction [7,9]; in particular this is the case of the Weinberg equation [12], where the

representation of angular momentum operators cannot be defined straightforwardly, since these operators would depend on interaction in a very complicated way. Furthermore, as stressed by Leutwyler and Stern [7], without defining angular momentum, the wave function cannot be normalized in a usual way. Therefore the normalization condition for the wave-function satisfying the Weinberg equation is not clear, except in ZRA, where the S-wave alone is involved.

However, as shown by Terent'ev [13], in the two-body sector with a finite range of interaction it is also possible to choose for the angular momentum operators the same representation as for free particles. For example, in the relativistic oscillator model [22] the two different representations are related by a "gauge" transformation [9]. Usually one assumes this equivalence to hold true also for other interactions. In the Terent'ev representation the angular momentum operator takes the form

$$\vec{\mathbf{l}} = -i\vec{\mathbf{q}} \times \frac{\partial}{\partial\vec{\mathbf{q}}}, \tag{2.21}$$

where

$$\vec{\mathbf{q}} \equiv (\mathbf{q}_T, q_z)\,, \qquad q_z = (\xi - \frac{1}{2})M_{12} - \frac{m_1^2 - m_2^2}{2M_{12}}, \tag{2.22}$$

$$M_{12} = M_{12}(\vec{\mathbf{q}}) = \epsilon_1 + \epsilon_2, \tag{2.23}$$

$$\epsilon_i = \sqrt{\vec{\mathbf{q}}^2 + m_i^2}, \quad i = 1, 2 \tag{2.24}$$

and the variables ξ and $\mathbf{q}_\mathrm{T}$ are defined by eqs. (2.19). In this representation angular momentum conservation is ensured by rotation invariance of the kernel in eq. (2.16).

Then it is possible to rewrite eq. (2.16) in the form

$$\hat{M}_{12}^2 \Psi = M^2 \Psi\ , \tag{2.25}$$

where

$$\hat{M}_{12}^2 = M_{12}^2(\vec{\mathbf{q}}) + W_{12}, \tag{2.26}$$

and

$$W_{12} = iU_{12}. \tag{2.27}$$

If W_{12} commutes with the angular momentum operator, the angular condition is satisfied and a covariant [7] description of the composite system is guaranteed.

Extension to the case or two interacting spinning particles can be performed in a straightforward way by using the Terent'ev representation [13]. In this case the meson state vector can be written in the form

$$| \; P_+, \mathbf{P}_T, J, \lambda >_{NP} =$$

$$\sum_{ln\sigma} \sum_{\nu'\mu'\nu\mu} C^{1\lambda}_{ln,1\sigma} C^{1\sigma}_{\frac{1}{2}\mu,\frac{1}{2}\nu} \int \frac{d^2 q_T}{P_+} \frac{d\xi}{\xi} \phi_l(q) Y_{ln}(\hat{q}) \; \hat{V}_{\mu'\mu} \; (\vec{\mathbf{q}}) \hat{V}_{\nu'\nu}(-\vec{\mathbf{q}}) | p_1 \mu', P - p_1 \nu' >_{NP},$$

where NP means "null plane", the C's are Clebsch-Gordan coefficients, Y_{ln} the Legendre functions, $\vec{\mathbf{q}}$ is defined by eq. (2.22), $p_1 = p_1(\vec{\mathbf{q}})$ is the four-momentum of the quark,

$$q = | \; \vec{\mathbf{q}} \; |, \qquad \hat{q} = \frac{\vec{\mathbf{q}}}{q} \tag{2.28}$$

and the $\hat{V}$'s are the Melosh [23] rotation matrices for quark and antiquark, i. e. [13],

$$\hat{V}(\vec{\mathbf{q}}) = \frac{m_1 + \epsilon_1 + q_z + i\epsilon_{js}\sigma_j q_s}{[2(\epsilon_1 + m_1)(\epsilon_1 + q_z)]^{\frac{1}{2}}} \tag{2.29}$$

$$\hat{V}(-\vec{\mathbf{q}}) = \frac{m_2 + \epsilon_2 - q_z - i\epsilon_{js}\sigma_j q_s}{[2(\epsilon_2 + m_2)(\epsilon_2 - q_z)]^{\frac{1}{2}}} \tag{2.30}$$

The state vector $| \; P_+, \mathbf{P}_T, J, \lambda >_{NP}$ at four-momentum P is obtained from the state vector at rest $| \; P_{0+}, \mathbf{0}_T, J, \lambda >_{NP}$ by the element of the Poincaré group $l(P \leftarrow P_0)$ equivalent

to the product of two Lorentz transformations, i. e., $\lambda(P \leftarrow P_\infty)\ \lambda(P_\infty \leftarrow P_0)$, where $P_0 \equiv (M,0,0,0)$.

Now let us consider nucleons interacting through exchange of different mesons. The kernel that describes such an interaction can be identified with the Born term expressed in terms of null - plane variables. For example, in the cases of scalar and vector meson exchange, the kernels have, respectively, the following forms [14]:

$$W_{12s,v} = \mathcal{V}^{+}(\vec{\mathbf{q}}')W^{(r)}_{s,v}(\vec{\mathbf{q}}',\ \vec{\mathbf{q}})\mathcal{V}(\vec{\mathbf{q}}), \tag{2.31}$$

where

$$\mathcal{V}(\vec{\mathbf{q}}) = \hat{V}(p_1)\otimes\hat{V}(p_2)\,, \qquad \mathcal{V}(\vec{\mathbf{q}}') = \hat{V}(p_1')\otimes\hat{V}(p_2')\,, \tag{2.32}$$

$$W^{(r)}_{12s}(\vec{\mathbf{q}}',\vec{\mathbf{q}}) = \frac{g^2}{(\vec{\mathbf{q}}' - \ \vec{\mathbf{q}})^2+\mu^2}\overline{u}(p_1')u(p_1)\ \overline{u}(p_2')u(p_2), \quad p_i^{(')} \equiv (\epsilon_i^{(')},\ (-)^{1+i}\vec{\mathbf{q}}^{(')}), \tag{2.33}$$

$$(W^{(r)}_{12v})_{\alpha'\beta',\alpha\beta}(\vec{\mathbf{q}}',\vec{\mathbf{q}}) \ = \frac{g^2}{(\vec{\mathbf{q}}' - \ \vec{\mathbf{q}})^2+\mu^2}\ V^{\mu}_{\alpha\beta}(\vec{\mathbf{q}}',\ \vec{\mathbf{q}})V^{\nu}_{\alpha\beta}(-\vec{\mathbf{q}}',-\vec{\mathbf{q}})\ g_{\mu\nu}, \tag{2.34}$$

$$V^{i}_{\alpha\beta}(\vec{\mathbf{q}}',\vec{\mathbf{q}}) = (\vec{\mathbf{q}}' + \ \vec{\mathbf{q}})\delta_{\alpha\beta} + i\left[(\vec{\mathbf{q}}' - \vec{\mathbf{q}})\ \times\vec{\sigma}\right]_{\alpha\beta}, \quad (i=1,2), \tag{2.35}$$

$$V^{0}_{\alpha\beta}(\vec{\mathbf{q}}',\vec{\mathbf{q}}) = \frac{\left[(\epsilon+m)^2 + \vec{\mathbf{q}}\cdot\vec{\mathbf{q}}'\right]\ \delta_{\alpha\beta} + i\vec{\sigma}_{\alpha\beta}\cdot\vec{\mathbf{q}}\ \times\vec{\mathbf{q}}'}{\epsilon+m}, \tag{2.36}$$

μ is the mass of the exchanged meson and $\vec{\mathbf{q}}'$ is defined analogously to $\vec{\mathbf{q}}$ (eq. 2.22), with M_{12} substituted by M_{12}', which generally assumes a different value. In the case of vector meson exchange we have assumed the two interacting particles to have equal masses, $m_1 = m_2 = m$, therefore we have also $\epsilon_1 = \epsilon_2 = \epsilon$. In the Terent'ev representation of angular momentum, such kernels satisfy the angular condition [18].

If we consider kernels of this kind, there is no exact correspondence between null plane dynamics in two - body sector and field theory; however the former description may be a good approximation to the latter one if only two - body intermediate states are important. A further improvement would consist in constructing multichannel null - plane dynamics; indeed, if the number of channels becomes infinite, null - plane dynamics and field theory will turn out to be equivalent.

Equation (2.16) may be also written symbolically as

$$\Psi = G_0 W_{12} \Psi, \tag{2.37}$$

where G_0 is defined by eq. (2.12). This leads immediately to the Lippmann-Schwinger equation for the two body relativistic T - matrix, i. e.,

$$T = W_{12} + W_{12} G_0 T, \tag{2.38}$$

which, in momentum representation, reads

$$T(x_1, \mathbf{p}_{1T}, x_2, \mathbf{p}_{2T}; x'_1, \mathbf{p}'_{1T}, x'_2, \mathbf{p}'_{2T}) = W_{12}(x, \mathbf{p}_T; x', \mathbf{p}'_T) +$$

$$+ \int dL''_{12} \frac{W_{12}(x, \mathbf{p}_T; x'', \mathbf{p}''_T) T(x''_1, \mathbf{p}''_{1T}, x''_2, \mathbf{p}''_2; x'_1, \mathbf{p}'_{1T}, x'_2, \mathbf{p}'_{2T})}{M^2 + \mathbf{P}_T^2 - \frac{\mathbf{p}''^2_{1T} + m_1^2}{x''_1} - \frac{\mathbf{p}''^2_{2T} + m_2^2}{x''_2} + i\epsilon}. \tag{2.39}$$

As proved in ref. [13], in this scheme angular momentum operators are the same as for free particles, the only difference consisting in the appearence of Melosh rotation matrices in partial wave expansion of the wave-function and of the scattering amplitude.

III. THREE-BODY NULL PLANE EQUATION

Analogously to the two-body case, the reduced three-body amplitude reads

$$(p_1^2-m_1^2)(p_2^2-m_2^2)(p_3^2-m_3^2)\phi(p_1,p_2,p_3)+$$

$$+\int\frac{d^4p'_1}{(2\pi)^4}\frac{d^4p'_2}{(2\pi)^4}\frac{d^4p'_3}{(2\pi)^4}(2\pi)^4\delta^4(P-p'_1-p'_2-p'_3)U_{123}(\Pi,\Pi';P)\phi(p'_1,p'_2,p'_3)=0, \tag{3.1}$$

where U_{123} is the kernel and

$$\Pi\equiv(p_1,p_2,p_3),\qquad \Pi'\equiv(p'_1,p'_2,p'_3). \tag{3.2}$$

The null plane wave-function is defined by

$$\psi'(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)=\int dp_{1-}dp_{2-}dp_{3-}p_{1+}p_{2+}p_{3+}\delta(P_- -p_{1-}-p_{2-}-p_{3-})\phi(p_1,p_2,p_3)\ . \tag{3.3}$$

Now let us suppose the kernel to be independent of p_{r-}, i. e.,

$$U_{123}=U_{123}(\mathbf{\Pi},\mathbf{\Pi}';P), \tag{3.4}$$

where

$$\mathbf{\Pi}\equiv(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3),\quad \mathbf{\Pi}'\equiv(\mathbf{p}'_1,\mathbf{p}'_2,\mathbf{p}'_3). \tag{3.5}$$

In this case we derive from (3.1) the following equation for null plane three-body wave-function:

$$\psi'(\mathbf{\Pi})=-G'_0(x_i,\mathbf{p}_{iT})\int dL'_{123}U_{123}(\mathbf{\Pi},\mathbf{\Pi}';P)\psi'(\mathbf{\Pi}') \tag{3.6}$$

where

$$dL'_{123}=\frac{1}{4(2\pi)^6}\prod_{r=1}^{3}\frac{dx'_r}{x'_r}d\mathbf{p}'_{rT}\delta(1-x'_1-x'_2-x'_3)\delta^2(\mathbf{P}_T-\mathbf{p}'_{1T}-\mathbf{p}'_{2T}-\mathbf{p}'_{3T}) \tag{3.7}$$

and

$$G'_0(x_i,\mathbf{p}_{iT})=\left[\frac{\mathbf{p}_{1T}^2+m_1^2}{x_1}+\frac{\mathbf{p}_{2T}^2+m_2^2}{x_2}+\frac{\mathbf{p}_{2T}^2+m_2^2}{x_2}-M^2-\mathbf{P}_T^2-i\epsilon\right]^{-1}\ . \tag{3.8}$$

If we assume the kernel U_{123} to depend on p_{r+} only through fractional momenta x_r, the three-body equation reads

$$\left[\frac{\mathbf{p}_{1T}^2+m_1^2}{x_1}+\frac{\mathbf{p}_{2T}^2+m_2^2}{x_2}+\frac{\mathbf{p}_{3T}^2+m_3^2}{x_3}-M^2-\mathbf{P}_T^2\right]\Psi'(x_i,\mathbf{p}_{iT})+$$

$$+\int dL'_{123}U_{123}(x_r,\mathbf{p}_{rT};x'_s,\mathbf{p}'_{sT})\Psi'(x'_i,\mathbf{p}'_{iT})=0\ . \tag{3.9}$$

We write, as in the two-body case, the Lippmann-Schwinger equation for three body T-matrix, i. e.,

$$T(x_i,\mathbf{p}_{iT};x'_i,\mathbf{p}'_{iT})=U_{123}(x_r,\mathbf{p}_{rT};x'_s,\mathbf{p}'_{sT})+$$

$$+\int dL''_{123}\frac{U_{123}(x_r,\mathbf{p}_{rT};x'_s,\mathbf{p}'_{sT})T(x''_i,\mathbf{p}''_{iT};x'_i,\mathbf{p}'_{iT})}{M^2+\mathbf{P}_T^2-\frac{\mathbf{p}''^2_{1T}+m_1^2}{x''_1}-\frac{\mathbf{p}''^2_{2T}+m_2^2}{x''_2}-\frac{\mathbf{p}''^2_{3T}+m_3^2}{x''_3}+i\epsilon}. \tag{3.10}$$

Now we assume only two-body interactions to occur. Then the kernel of eq. (3.9) may be written as

$$U_{123}(x_r,\mathbf{p}_{rT};x'_s,\mathbf{p}'_{sT})=i\sum_{i\neq j\neq k}x_kU_{ij}(x_{ij},\mathbf{p}_{ijT};x'_{ij},\mathbf{p}'_{ijT})2(2\pi)^3\delta(x_k-x'_k)\delta^2(\mathbf{p}_{kT}-\mathbf{p}'_{kT}), \tag{3.11}$$

where $\mathbf{p}_{ijT}$ and x_{ij} are defined analogously to eq. (2.15). Then equation (3.9) can be written as

$$\hat{M}^2_{123}\Psi'=M^2\Psi', \tag{3.12}$$

where we have set

$$\hat{M}^2_{123}=\sum_{i=1}^{3}\frac{m_i^2+\mathbf{p}_{iT}^2}{x_i}+\sum_{i\neq j\neq k}\frac{W_{ij}}{1-x_k}, \tag{3.13}$$

$$W_{ij}\Psi'=i\int dL^{k'}_{ij}U_{ij}(x_{ij},\mathbf{p}_{ijT};x'_{ij},\mathbf{p}'_{ijT})\Psi'(\mathrm{x}'_\mathrm{i},\mathbf{p}'_{iT};x'_j,\mathbf{p}'_{jT};x_k,\mathbf{p}_{kT}) \tag{3.14}$$

and $dL_{ij}^{k'}$ is defined by (2.17), substituting the overall transverse momentum $\mathbf{P}_T$ by

$$\mathbf{P}_{ijT} = \mathbf{p}_{iT} + \mathbf{p}_{jT}. \tag{3.15}$$

In the case of ZRA W_{ij} are constants and condition (3.4) is satisfied automatically. Such an equation is equally satisfied if the two–body interactions are of the type discussed in the previous section. Here we show that eq. (3.9), with a kernel like (3.11), satisfies cluster separability condition. Indeed, if, e. g., the third particle is taken far away from the other two, we have $W_{23} = W_{31} = 0$ and eq. (3.9) reduces to

$$\left(\frac{\mathbf{P}_{12T}^2 + \hat{M}_{12}^2}{1 - x_3} + \frac{\mathbf{p}_{3T}^2 + m_3^2}{x_3} - \mathbf{P}_T^2\right)\Psi' = M^2\Psi', \tag{3.16}$$

where $\hat{M}_{12}^2$ is given by eq. (2.26).

Dividing both sides of eq. (3.16) by $2P_+$, we get

$$P_-\Psi' = (P_{12-} + p_{3-})\Psi', \tag{3.17}$$

which proves cluster separability for P_-.

In this connection let us note that the semi-relativistic three-body Hamiltonian

$$H = T(\vec{\mathbf{p}}_1, \vec{\mathbf{p}}_2, \vec{\mathbf{p}}_3) + V_{123}\left(\vec{\mathbf{p}}_1, \vec{\mathbf{p}}_2, \vec{\mathbf{p}}_3\right), \tag{3.18}$$

where T is the relativistic kinetic term,

$$T(\vec{\mathbf{p}}_1, \vec{\mathbf{p}}_2, \vec{\mathbf{p}}_3) = \sum_{i=1}^{3} \left[\vec{\mathbf{p}}_i^2 + m_i^2\right]^{\frac{1}{2}} \tag{3.19}$$

and the potential is assumed to be the sum of two-body interactions, i. e.,

$$V_{123} = \sum_{i<j} V_{ij}, \tag{3.20}$$

does not satisfy the cluster separability condition. Indeed, if one particle is taken far away, the Hamiltonian in the centre-of-mass system should have the form

$$H_{12+3}^{c.m.} = \sqrt{\vec{\mathbf{Q}}_{12}^2 + \hat{M}_{12}^2} + \sqrt{\vec{\mathbf{Q}}_{12}^2 + m_3^2}, \tag{3.21}$$

where $\vec{\mathbf{Q}}_{12}$ is the total momentum of the pair $(1-2)$.

On the other hand, in ref. [18] the three-body mass operator has been assumed to be of the form

$$H_{123}^{c.m.} = M_0 + \sum_{i<j} (\hat{E}_{ij} - E_{ij}) \tag{3.22}$$

where M_0 is the overall free mass operator,

$$E_{ij} = \sqrt{\vec{\mathbf{Q}}_{ij}^2 + M_{ij}^2} \qquad \hat{E}_{ij} = \sqrt{\vec{\mathbf{Q}}_{ij}^2 + \hat{M}_{ij}^2}, \tag{3.23}$$

and $\hat{M}_{ij}^2$, M_{ij}^2 are defined analogously to eqs. (2.26) and (2.23). Evidently the mass operator (3.22) satisfies the cluster separability condition (3.17). However, if we define the mass operator squared for the three-body system as

$$\hat{M}_{123}^2 = (H_{123}^{c.m.})^2, \tag{3.24}$$

this is different from eq. (3.13), derived from the BS equation.

It is worth noting that the mass operator (3.22), which was considered in [18], commutes with the angular momentum operators taken in the same representation as for free particles. This is not true for the mass operator defined by eq. (3.13). However we do not worry about this; indeed, when considering the baryon Regge trajectories in Sect. 5, we shall propose a different kernel, which correponds to a three-force with string junction and commutes with the free angular momentum operators.

We note also that eq. (3.13) has been considered in ref. [24], however the form of this equation has been postulated, no argument has been given for deriving it.

IV. MESON REGGE TRAJECTORIES

Now we apply two- and three-body equations to the description of meson and baryon Regge trajectories. In particular in this section we consider mesons. We relate the $q\overline{q}$ interaction term W_{12} (eq. (2.26)) in the two-body equation to the potential used by Godfrey and Isgur [3], i. e.,

$$V_{12} = V_{12}^{conf} + V_{12}^{hyp} + V_{12}^{so} \tag{4.1}$$

where

$$V_{12}^{conf} = \left[c + \nu r - \frac{4}{3}\ \frac{\alpha_s(r)}{r} \right]$$
$$c = -0.253 \quad GeV, \qquad \nu = 0.18 \quad GeV^2, \tag{4.2}$$

r is the distance between q and $\overline{q}$ in the two-body cms and V_{12}^{hyp} and V_{12}^{so} are, respectively, the hyperfine and spin-orbit interactions [3]. Squaring the mass operator

$$\hat{M}_{12} = M_{12} + V_{12}, \tag{4.3}$$

we recover eq. (2.26), with

$$W_{12} = M_{12}V_{12} + V_{12}M_{12} + V_{12}^2. \tag{4.4}$$

Obviously in this case eq. (2.26) provides the same spectrum as Godfrey and Isgur [3].

Here we concentrate our attention on Regge trajectories for systems of light quarks. For large quark-antiquark separations the hyperfine and spin-orbit interactions (except for the Thomas precession term) can be neglected and the main contribution comes from the confining term. On the other hand, we know from experiment that Regge trajectories are to

a high extent linear, with the same slope. For large orbital excitations a linear l-dependence (where l is the orbital angular momentum) of squared masses, i. e.,

$$M^2 = 8\nu l \tag{4.5}$$

can be found by using a linear confining potential in the mass operator. Now we pose the question how to reproduce phenomenologically such a linearity in the framework of the two-body equation (2.25). If we assume W_{12} to be of the form

$$W_{12}(r) = \omega^2 r^\zeta, \tag{4.6}$$

then for massless constituents and for orbital angular momenta $l >> 1$ we may consider the semiclassical limit for M^2, i. e.,

$$M^2 = 4\frac{l^2}{r^2} + \omega^2 r^\zeta \ . \tag{4.7}$$

The mass squared of the system has a minimum when r satisfies the equation

$$\frac{dM^2}{dr} = 0, \tag{4.8}$$

whose solution is

$$r = (\frac{8l^2}{\zeta\omega^2})^{\frac{1}{\zeta+2}}. \tag{4.9}$$

In this case we have the following Regge trajectory:

$$M^2 = 4l^2(\frac{\zeta\omega^2}{8l^2})^{\frac{2}{\zeta+2}} + \omega^2(\frac{8l^2}{\zeta\omega^2})^{\frac{\zeta}{\zeta+2}}, \tag{4.10}$$

which turns out to be linear when $\zeta = 2$. In this case

$$M^2 = 4\omega l. \tag{4.11}$$

Comparing eq. (4.11) with eq. (4.5), we find $\omega = 2\nu$.

This means that for large $q - \overline{q}$ separations the kernel W_{12} is of the oscillator form. Taking into account that all the Regge trajectories are approximately linear, with the same slope, even for low l, (see, for example, [26,27]), we can set the kernel (4.4) in the form

$$W_{12} = W_0 + 4\nu^2 r^2 + \Delta W_{12}(r)\,, \tag{4.12}$$

where $\Delta W_{12}(r)$ - which generates non-linear corrections to Regge trajectories, e. g., spin dependent terms - is expected to be small.

We apply eq. (2.25) with the kernel (4.12) to the description of $q\overline{q}$ Regge trajectories, where $q = u, d$ or s. As it follows from the calculations of $q\overline{q}$ meson spectrum done by Godfrey and Isgur [3], spin-dependent corrections are quite important for $1s$ and $2s$-states. For $1p$-states they should also be taken into account: the splitting between 1^3P_2 and 1^3P_0 levels is about 200 MeV. However for all other states those corrections are small and can be neglected (see also the discussion at the end of this section). If we neglect $\Delta W_{12}(r)$, the spectra of mesons composed of u and d quarks are given by

$$M^2_{nl} = 4m^2 + W_0 + 4\omega[2n + l - 1/2] \quad (n \geq 1)\,. \tag{4.13}$$

We take the same masses of u and d (constituent) quarks as in ref. [3], i.e. ,

$$m = m_u = m_d = .22 \text{ GeV}. \tag{4.14}$$

The two parameters W_0 and ω have been fixed fitting the ρ-trajectory (see Fig. 1(a)), i.e.,

$$W_0 = -1.255 \quad \text{GeV}^2, \qquad \omega = 0.2836 \quad \text{GeV}^2. \tag{4.15}$$

There are eight meson Regge trajectories made of u and d quarks. They are degenerate in isospin. For each isospin I there are three trajectories with angular momenta $j = l + 1$,

$j = l$ and $j = l - 1$ and total quark-antiquark spin $s_{12} = 1$ and one trajectory with $j = l$ and $s_{12} = 0$. We use the notation

$$M^I_{l=j-1,s_{12}=1}, \quad M^I_{l=j,s_{12}=1}, \quad M^I_{l=j+1,s_{12}=1}, \quad M^I_{l=j,s_{12}=0}\,. \tag{4.16}$$

All such trajectories are plotted in Figs. 1. Solid lines are determined using eq. (4.13). We observe that all available experimental data are in good agreement with the spectrum given by eq. (4.13) for $l \geq 1$. Approximate linearity with equal slopes of all nonstrange meson Regge trajectories for $l \geq 1$ confirms *a posteriori* that nonlinear, spin dependent terms of the potential fall down rapidly at increasing l. At $l = 0$ these contributions have to be taken into account, as appears in the plots. Such deviations are smallest in the ρ-trajectory, suggesting that in this case nonlinear effects are less important.

When the masses of the quark and antiquark are different, we represent the mass operator squared in the form

$$\hat{M}^2_{12} = 4(\alpha\vec{\mathbf{q}}^2 + \mu^2) + W_{12} + \; G(\alpha, \mu^2, \vec{\mathbf{q}}^2)\,, \tag{4.17}$$

where

$$\mu = \frac{m_1 + m_2}{2} \tag{4.18}$$

and

$$G(\alpha, \mu^2, \vec{\mathbf{q}}^2) = (\sqrt{\vec{\mathbf{q}}^2 + m_1^2} + \sqrt{\vec{\mathbf{q}}^2 + m_2^2})^2 - 4(\alpha\vec{\mathbf{q}}^2 + \mu^2)\,. \tag{4.19}$$

Note that G is equal to 0 when

- $m_1 = m_2$ and $\alpha = 1$;
- $\vec{\mathbf{q}} \to 0$, for any value of m_1 and m_2 ;

- $\vec{\mathbf{q}}^2 >> m_1^2, m_2^2$ and $\alpha = 1$.

If $m_1 \neq m_2$, the eigenvalues and eigenfunctions of the operator (4.17) can be suitably approximated by solving the equation

$$\left[4\left(\alpha\vec{\mathbf{q}}^2 + \mu^2\right) + W_{12}\right]\Psi = M^2\Psi\,, \tag{4.20}$$

where W_{12} is given by eq. (4.12), with $\Delta W_{12} = 0$. We choose α in such a way that G corrections to eigenvalues of (4.20) vanish to first order. This can be realized to a good approximation by imposing α to be a solution of the equation

$$G(\alpha, \mu^2, \overline{\vec{\mathbf{q}}^2}(\alpha)) = 0\,, \tag{4.21}$$

where $\overline{\vec{\mathbf{q}}^2}(\alpha)$ is defined as the average value of $\vec{\mathbf{q}}^2$ over the wave function Ψ, which obviously depends itself from α. As can be checked, eq. (4.21) is an algebraic equation of the third degree in $\sqrt{\alpha}$; among the roots of this equation we pick up the one which tends to 1 as $m_1 - m_2$ tends to 0. Let us examine the solution in detail. The eigenfunctions of eq. (4.20) are

$$\Psi_{nlm}(\vec{\mathbf{r}}) = \phi_{nl}(r)Y_l^m(\theta,\phi), \tag{4.22}$$

$$\phi_{nl} = C_{nl}e^{-1/2\beta^2 r^2}(\beta r)^l F\left(1-n, l+\frac{3}{2}, \beta^2 r^2\right), \tag{4.23}$$

$$C_{nl} = \frac{\beta^{l+3/2}}{\Gamma(l+3/2)}\sqrt{\frac{2\Gamma(n+l+1/2)}{\Gamma(n)}}, \tag{4.24}$$

$$\beta^2 = \frac{\omega}{2\sqrt{\alpha}}. \tag{4.25}$$

$\overline{\vec{\mathbf{q}}^2}(\alpha)$ depends on n, l:

$$\overline{\vec{\mathbf{q}}^2}(\alpha) =< nlm|\vec{\mathbf{q}}^2|nlm >= (n + l/2 - 1/4)\,\frac{\omega}{\sqrt{\alpha}}\,, \tag{4.26}$$

having denoted by $|nlm>$ the eigenstates of eq. (4.20). Therefore the solution of eq. (4.21) which we have picked up also depends on n, l through the combination $g_{nl} = 2n + l - 1/2$. Then the eigenvalues of equation (4.17) can be approximated by the following expression:

$$M_{nl}^2 = 4\mu^2 + W_0 + 4\,(2n + l - 1/2)\,\omega\sqrt{\alpha_{nl}}\,. \tag{4.27}$$

The plot of α_{nl} as a function of g_{nl} is shown in Fig. 2 for $m_1 = .22$ GeV and m_2 ranging from .22 to 1.22 GeV. When $m_2 \leq .7$ GeV, the difference between α_{nl} and 1 does not exceed 10%. In this case the correction to the slope of the Regge trajectory does not exceed 5% and the deviations from linearity are negligible.

We apply (4.27) to the description of the strange meson trajectories. For the mass of the strange quark we used the same value as in ref. [3], i.e. $m_s = .419$ GeV. Then we can predict all the strange meson trajectories ($M_{l,s_{12}}^{1/2}$) and mesons with hidden strangeness ($M_{l,s_{12}}^{0}$) - see Figs. 3 and 4.

As in the case of non-strange mesons, the agreement between our theoretical predictions and the existing experimental data [28] is good, except for the states with $l = 0$.

The spectra given by eq. (4.27) contain also radial excitations. The mass of the $q\overline{q}$ state with $n = 2,\ l = 0$ is predicted around 1.7 GeV, while the in pseudoscalar channel the first radial excitation is around 1.3 GeV ($\pi(1300)$ and $\eta(1295)$) and in the vector channel it is around 1.4 GeV ($\omega(1390)$ and $\rho(1450)$). This discrepancy can be due to neglecting, in our model, spin-spin interaction, which mixes states with different n.

V. BARYON REGGE TRAJECTORIES

Let us apply now the three-body null plane equation (3.13) to the description of the baryon Regge trajectories. We consider a system of three light quarks in a colour singlet

state. For orbital excitations with large angular momenta we may distinguish among three different configurations:

1. String-like configuration, with two quarks forming a diquark ($r_{12} \sim r_{23} \gg r_{31}$, where r_{ij} is the relative distance between the $i-$th and the $j-$th quark in the two-particle cms) ["D"-configuration];

2. Symmetric triangle-like configuration ($r_{12} \sim r_{23} \sim r_{31}$) ["T"-configuration];

3. Star-like configuration with string junction ["Y"-configuration].

In the first case, the problem reduces to the two-body case; in particular ω is the same as in quark-antiquark system, since in a baryon the diquark is in a colour anti-triplet state. For rather large overall angular momenta, when the rest mass of the diquark may be neglected, the Regge trajectory is the same as in two-body case, eq. (4.11).

In the second case we recover eq. (3.13), where the two-body kernels W_{ij} are of the oscillator form. Taking into account the colour degree of freedom yields

$$W_{ij} = \frac{1}{2}\omega^2 r_{ij}^2. \tag{5.1}$$

If, moreover, we take into account of the symmetry of the configuration and proceed similarly to the two body case, we find

$$M^2 = 6\omega L_{123}\,, \tag{5.2}$$

where L_{123} is the overall angular momentum of the three-quark system. Therefore for a fixed L_{123} the diquark-quark configuration has a lower energy than the symmetric triangle configuration, according to the following ratio:

$$M_T^2 : M_D^2 = 1.5 : 1\,. \tag{5.3}$$

In the case of a relativized three-body Schrödinger equation, i. e.,

$$(\sum_{i=1}^{3} \sqrt{\vec{\mathbf{p}}_i^2 + m_i^2} + V_{123})\Psi' = M\Psi', \tag{5.4}$$

where V_{123} is linear either with respect to relative distances between quarks (triangle configuration) or with respect to distances of quarks from the string centre (string junction configuration), the situation is different; in this case it results [29]

$$M_T^2 : M_D^2 = 1.3 : 1.0. \tag{5.5}$$

The difference in the ratio $M_T^2 : M_D^2$ is due to the relativistic recoil effects taken into account in the three-body null plane equation.

In connection with triangle configuration we recall, as shown in sect. 3, that the three-body semi-relativistic Hamiltonian with a two-body potential does not satisfy cluster separability condition. Alternatively, using three-body mass operator (3.22) - which satisfies cluster separability condition - would lead to a non-linear Regge trajectory. In fact, if we assume the potential V_{ij}, which describes quark-quark interaction in the two-quark cms, to be proportional to the distance between the two interacting particles - as imposed by linearity of meson Regge trajectories -, the same potential in the overall cms is no longer linear, since, according to eqs. (3.22) and (3.23), it transforms to

$$V'_{ij} = \hat{E}_{ij} - E_{ij} \tag{5.6}$$

where

$$\hat{E}_{ij} = \sqrt{\vec{\mathbf{Q}}_{ij}^2 + (M_{ij} + V_{ij})^2}\,, \qquad E_{ij} = \sqrt{\vec{\mathbf{Q}}_{ij}^2 + M_{ij}^2} \tag{5.7}$$

and M_{ij} is the two-body free mass operator, defined analogously to (2.23).

In the case of a star-like configuration with string junction the interaction is not a two body one. This is a genuine three-body force. Indeed arguments in favour of a three-body force for describing the baryon spectrum have been illustrated in ref. [30]. Also in the framework of a relativized Schrödinger equation a three-body force with string junction [4] allows a unified treatment of mesons and baryons, whereas exclusive use of two-body forces [6] demands a quark-diquark structure. Then we assume a kernel of the form

$$U_{123} = U_0 + \omega'^2 (r_{01}^2 + r_{02}^2 + r_{03}^2), \qquad r_{0i} = |\vec{\mathbf{r}}_{0i}|, \qquad (i = 1, 2, 3), \tag{5.8}$$

where U_0 is a constant and $\vec{\mathbf{r}}_{0i}$ is the vector from the string center to the $i-$th quark in the overall cms. It is not difficult to recognize that such a kernel is of the type (3.4), since $\vec{\mathbf{r}}_{0i}$ is conjugate to momentum

$$\vec{\mathbf{p}}_{0i} = \vec{\mathbf{p}}_i - \vec{\mathbf{p}}_0, \tag{5.9}$$

where

$$\vec{\mathbf{p}}_l \equiv (\mathbf{p}_{lT}, p_{lz}) \qquad (l = 0, 1, 2, 3) \tag{5.10}$$

and the p_{lz} are related to the fractional momenta ξ_l by means of the following equations:

$$\xi_l = \frac{\sqrt{m_l^2 + \vec{\mathbf{p}}_l^2} + p_{lz}}{M}, \qquad (l = 0, 1, 2, 3). \tag{5.11}$$

The kernel (5.8) contains also the dependence on c.m. coordinate. To eliminate it, we proceed as in the non relativistic case of three particles with equal masses, defining the relative coordinates as follows:

$$\vec{R} = \frac{1}{3} (\vec{r_{01}} + \vec{r_{02}} + \vec{r_{03}})$$

$$\vec{\rho} = \vec{r_{01}} - \vec{r_{02}} \tag{5.12}$$
$$\vec{\lambda} = \frac{1}{3}\left(\vec{r_{01}} + \vec{r_{02}} - 2\vec{r_{03}}\right) .$$

In order to take into account relativistic effects, we write the interaction term in the form

$$U_{123} = U_0 + {\omega'}^2 \left(\frac{2}{3}\eta_\lambda \vec{\lambda}^2 + \frac{1}{2}\eta_\rho \vec{\rho}^2\right) . \tag{5.13}$$

In the non relativistic limit the factors η_λ and η_ρ tend to 1. Let us stress a difference between our model and the non-relativistic harmonic oscillator model. In our case the interaction term (5.13) appears in the mass operator squared, i. e.,

$$\hat{M}^2_{123} = \left[\sqrt{\vec{\mathbf{Q}}^2 + M^2_{12}(\vec{\mathbf{q}}^2)} + \sqrt{\vec{\mathbf{Q}}^2 + m_3^2}\ \right]^2 + U_{123}, \tag{5.14}$$

(we have used shortened notation $\vec{\mathbf{Q}}$ for $\vec{\mathbf{Q}}_{12}$), while in the non relativistic case it is a part of the mass operator. Therefore the mass spectrum and the Regge trajectories in the relativistic and non relativistic models with harmonic oscillator interaction will be essentially different for systems with light quarks. The momenta $\vec{\mathbf{Q}}$ and $\vec{\mathbf{q}}$ in equation (5.14) are, respectively, the momenta of the subsystems $(12) + 3$ and $1 + 2$ in their c.m. frames.

In the nonrelativistic case $\vec{\lambda}$ and $\vec{\rho}$ are the conjugate coordinates, respectively, to the momenta $\vec{\mathbf{Q}}$ and $\vec{\mathbf{q}}$. In the relativistic case this can be justified only for $\vec{\lambda}$. Of course the value of $|\vec{\lambda}|$ is not Lorentz invariant and in the overall c.m. system it can be regarded as the distance between the subsystem $(1 + 2)$ and particle 3. In line with these considerations we can take the coefficient η_λ in eq. (5.13) equal to 1. The parameter η_ρ, which describes the relativistic recoil effect for subsystem $1 + 2$, will be fixed later. Therefore in the relativistic case the choice of the kernel (5.14), where both $\vec{\lambda}$ and $\vec{\rho}$ are considered as conjugate coordinates to the momenta $\vec{\mathbf{Q}}$ and $\vec{\mathbf{q}}$ should be treated as an "ansatz", which however does not correspond to

a sum of two-body interactions. An important consequence of this choice of the interaction is that it commutes with the angular momentum operators taken in the same representation as for free particles:

$$\vec{L}_{123} = \vec{L} + \vec{l} = -i\vec{\mathbf{Q}} \times \frac{\partial}{\partial \vec{\mathbf{Q}}} - i\vec{\mathbf{q}} \times \frac{\partial}{\partial \vec{\mathbf{q}}}. \tag{5.15}$$

Since the mass and angular momentum operators have been fixed according to eqs. (5.13) and (5.15) respectively, all other Poincaré generators can be constructed according to the scheme prposed by Berestetsky and Terent'ev [31] (see also [18]). The angular condition is satisfied, therefore the eigenstates of our Hamiltonian are guaranteed to be the null plane restrictions of covariant wavefunctions [7].

Let us consider the spectrum predicted by the mass operator squared (5.14). Using the results of the previous section, we write the kinetic term in eq. (5.14) as follows:

$$\left[\sqrt{\vec{\mathbf{Q}}^2 + M_{12}^2(\vec{\mathbf{q}}^2)} + \sqrt{\vec{\mathbf{Q}}^2 + m_3^2}\ \right]^2 = 4(\alpha\vec{\mathbf{Q}}^2 + \mu^2) + G(\alpha, \mu^2, \vec{\mathbf{Q}}^2), \tag{5.16}$$

$$\mu = \frac{1}{2}\left[M_{12}(\vec{\mathbf{q}}) + m_3\right], \tag{5.17}$$

and we assume $m_1 = m_2 = m$. As in the two-body case, we approximate the eigenvalues and eigenfunctions of the operator $\hat{M}_{123}^2$(eq. (5.14)) with those of the eigenvalue equation

$$\left[4(\alpha\vec{\mathbf{Q}}^2 + \mu^2) + \omega'^2\left(\frac{2}{3}\vec{\lambda}^2 + \frac{1}{2}\eta_\rho\vec{\rho}^2\right) + U_0\right]\Psi = M^2\Psi\,, \tag{5.18}$$

where α solves the equation

$$G(\alpha, \mu^2(\overline{\vec{\mathbf{q}}^2}), \overline{\vec{\mathbf{Q}}^2}) = 0\,. \tag{5.19}$$

Here $\overline{\vec{\mathbf{q}}^2}$ and $\overline{\vec{\mathbf{Q}}^2}$ are the average values of $\vec{\mathbf{q}}^2$ and $\vec{\mathbf{Q}}^2$ for the eigenfunctions of eq. (5.18), which can be written in the form

$$\Psi_{NLM,nlm}\left(\vec{\mathbf{Q}},\vec{\mathbf{q}}\right)=\psi_{NLM}\left(\vec{\mathbf{Q}}\right)\phi_{nlm}\left(\vec{\mathbf{q}}\right)\,,\qquad (N,n\geq 1) \tag{5.20}$$

where the functions $\psi_{NLM}\left(\vec{\mathbf{Q}}\right)$ and $\phi_{nlm}\left(\vec{\mathbf{q}}\right)$ are defined by the expressions (4.22-4.25), with

$$\beta_Q^2=\frac{1}{\sqrt{6}}\frac{\omega'}{\sqrt{\alpha}}\,,\qquad \beta_q^2=\frac{\omega'}{2\sqrt{2}}\sqrt{\eta_\rho}. \tag{5.21}$$

Assuming $\overline{M_{12}^2(\vec{\mathbf{q}})}>>m_3^2$, and moreover that $U_0=2W_0$, where W_0 is the same as for $q\overline{q}$ system (which amounts to taking W_0 per degree of freedom), we can write the eigenvalues of eq. (5.18) as follows:

$$M_{NLnl}^2=M_{nl}^2+W_0+4\omega'\sqrt{\frac{2}{3}\alpha_{NL}}\,(2N+L-1/2)\,, \tag{5.22}$$

where

$$M_{nl}^2=W_0+4m^2+2\sqrt{2}\omega'\sqrt{\eta_\rho}(2n+l-1/2)\,. \tag{5.23}$$

Let us consider the limiting configuration of $L\to\infty$ and l fixed, which corresponds to a string-like diquark-quark configuration. In this case, since $\vec{\mathbf{Q}}^2>>M_{12}^2,m_3^2$ and $\alpha_{NL}\to 1$, the slope of the Regge trajectory for this configuration results to be $\sqrt{3/2}(4\omega')^{-1}$; this slope equals the slope of the meson trajectory, therefore

$$\omega'=\sqrt{3/2}\omega\,. \tag{5.24}$$

On the other hand, in the limit $l\to\infty$, with L fixed, the slope of the Regge trajectory results to be $(2\sqrt{3}\omega\sqrt{\eta_\rho})^{-1}$. As the two slopes are equal, η_ρ is uniquely fixed to 4/3. Therefore the mass spectrum (5.22) has the form

$$M_{NLnl}^2=\{-2W_0+4m^2+4\omega[\sqrt{\alpha}(2N+L)+2n+l\ -1/2(1+\sqrt{\alpha})]\}\quad GeV^2\,, \tag{5.25}$$

where W_0, m and $\omega$are the same as for $q\overline{q}$ sector (see eqs (4.14) and (4.15)). After symmetrization over all quarks the wave-function (5.20) can be considered a good approximation

to the ground state of the Δ-isobar. According to eq. (5.25), the mass of the ground state ($N = n = 1$, $L = l = 0$) in this case is equal to $M_{1010} = 1.043$ GeV, which is intermediate between the masses of the nucleon and of the Δ-isobar. The spin-spin interaction will increase the mass for a total angular momentum $J = 3/2$ and decrease it for $J = 1/2$.

In the description of Regge trajectories M_{nl} can be considered as the effective mass of a diquark. According to eq. (5.23) for $n = 1$ and $l = 0$ we have $M_{10} = 0.8$ GeV. This value can be considered as an estimation of the mass of a diquark in the ground state at $n = 1$ and $l = 0$. To take into account spin-spin interaction we shall fit the baryon Regge trajectories considering the mass of the diquark as a free parameter. As we have already seen for mesons, the spin-spin interaction is particularly important in the s-wave. Therefore, if we consider the states with $l = 0$ and $L \geq 1$, the spin-spin interaction between the third quark and the two others may be neglected.

Let us classify the baryon Regge trajectories composed of u, d quarks in the diquark-quark picture. The lowest states are composed of s-wave diquarks: $\mathcal{D}_{00}(I_{12} = s_{12} = 0)$ and $\mathcal{D}_{11}(I_{12} = s_{12} = 1)$. There are two nucleon Regge trajectories corresponding to the orbital excitations of the $q - \mathcal{D}_{00}$ system with $I = 1/2$: we denote them by $N_{L=J-1/2}$ and $N_{L=J+1/2}$. Six Regge trajectories correspond to the orbital excitation of the $q - \mathcal{D}_{11}$ system. They are degenerated in the isospin $I = 1/2, 3/2$: $(N - \Delta)_{L=J-3/2}$, $(N - \Delta)_{L=J+1/2}$, $(N - \Delta)'_{L=J+1/2}$ and $(N - \Delta)_{L=J+3/2}$. The trajectories $(N - \Delta)$ and $(N - \Delta)'$ correspond to different total spins of quarks, $S_{123} = 3/2$ or $1/2$. To describe all these trajectories, we use the two body equation (4.27), where the mass of the quark was taken as for meson trajectories, i.e., $m_u = m_d = .22$ GeV and the masses of $\mathcal{D}_{00}$ and $\mathcal{D}_{11}$ diquarks have been found from fits to the main N and Δ trajectories, $N_{L=J-1/2}$ and $\Delta_{L=J-3/2}$ (see Figs. 5 and 6), yielding

$M_{00} = .44$ GeV and $M_{11} = .80$ GeV, which are consistent with the values given above.

The agreement of the theoretical predictions with the available experimental data is quite satisfactory. We are able also to predict the Λ Regge trajectory composed of $s - \mathcal{D}_{00}$ orbital excitations. Our predictions (see Fig. 6) are in very good agreement with data.

VI. CONCLUSIONS

Starting from the BS equation in ZRA, we have introduced a null plane wave-function and have derived a three-dimensional two- body equation. We have generalized this equation to the case of finite range interactions and have discussed the choice of angular momentum operators in this scheme. We have considered also the three-body problem in the same approach, showing that in ZRA it is possible to derive a three-body null plane Hamiltonian with two-body interaction which satisfies cluster separability condition. We have generalized also this approach to the case of two- and three-body interactions with finite range. As an application of this scheme we have considered Regge trajectories for relativistic two- and three-quark systems. We have shown that linear trajectories can be derived when the interaction term in the mass square operator is of the oscillator form.

We have discussed the relation between this kernel and the Hamiltonian used by Godfrey and Isgur [3]. Then we have applied our model to the description of the meson Regge trajectories $\bar{q}-q$, $\bar{q}-s$ ($\bar{s}-q$) and $\bar{s}-s$. The model describes, with only four free parameters, $m_u = m_d = m$, m_s, W_0 and ω, all orbitally excited meson states in a quite satisfactory way. We have considered also the choice of the kernel in the three-body case. It appear that the choice of two-body forces in the relativistic three-body equation satisfying the cluster separability property leads to an equation which is essentially different from the

semi-relativistic generalizations of the Schrödinger equation.

We have also considered a kernel which corresponds to a three-body force with string junction; this operator can be taken in a form which commutes with the angular momentum operator in the free particle representation, so that the angular condition is satisfied. Choosing this model of interaction, we consider Regge trajectories for three-quark systems. We calculate all baryon Regge trajectories composed of u and d quarks in the diquark-quark configuration introducing phenomenologically hyperfine splitting through the difference in masses of the two color antitriplet diquarks, $\mathcal{D}_{00}$ and $\mathcal{D}_{11}$. Our predictions are in good agreement with N and Δ Regge trajectories. Also the prediction of the Λ Regge trajectory, which corresponds to $s - \mathcal{D}_{00}$ orbital excitations, is in very good agreement with data.

To summarize, the approximation proposed for the BS equation or for the four-dimensional three-body equation turns out to be a covariant Hamiltonian dynamics [7], provided we consider a convenient representation of the angular momentum operators and choose a kernel which commutes with such operators. Moreover, concerning quark dynamics, this approximation may be implemented by three-body interactions with string junction - whose importance has been already stressed in the literature - to yield a unified description of meson and baryon Regge trajectories.

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FIGURE CAPTIONS

[Fig. 1] - Meson Regge trajectories made of u- and/or d-quarks (M^I_{L,s_12}).

[Fig. 2] - The parameter α_{nl} as a function of $g_{nl} = 2(n-1)+l$, found from the solution of eq. (4.21) for $m_1 = .22$ GeV and m_2 ranging from 0.22 to 1.22 GeV.

[Fig. 3] - Regge trajectories for strange mesons ($M^{1/2}_{L,s_12}$).

[Fig. 4] - Regge trajectories for $\overline{s}s$ mesons (M^0_{L,s_12}).

[Fig. 5] - Non-strange baryon Regge trajectories.

[Fig. 6] - Λ Regge trajectory, corresponding to the orbital excitations of the $s - \mathcal{D}_{\prime\prime}$ system.